Time Series Analysis & Forecasting Using R

10. Forecast reconciliation

bit.ly/fable2023

Outline

Outline

Australian Pharmaceutical Benefits Scheme

PBS sales

```
PBS
```

```
# A tsibble: 67,596 x 9 [1M]
# Key:       Concession, Type, ATC1, ATC2 [336]
      Month Concession   Type   ATC1  ATC1_desc ATC2  ATC2_desc Scripts
      <mth> <chr>        <chr>  <chr> <chr>     <chr> <chr>       <dbl>
 1 1991 Jul Concessional Co-pa~ A     Alimenta~ A01   STOMATOL~   18228
 2 1991 Aug Concessional Co-pa~ A     Alimenta~ A01   STOMATOL~   15327
 3 1991 Sep Concessional Co-pa~ A     Alimenta~ A01   STOMATOL~   14775
 4 1991 Oct Concessional Co-pa~ A     Alimenta~ A01   STOMATOL~   15380
 5 1991 Nov Concessional Co-pa~ A     Alimenta~ A01   STOMATOL~   14371
 6 1991 Dec Concessional Co-pa~ A     Alimenta~ A01   STOMATOL~   15028
 7 1992 Jan Concessional Co-pa~ A     Alimenta~ A01   STOMATOL~   11040
 8 1992 Feb Concessional Co-pa~ A     Alimenta~ A01   STOMATOL~   15165
 9 1992 Mar Concessional Co-pa~ A     Alimenta~ A01   STOMATOL~   16898
10 1992 Apr Concessional Co-pa~ A     Alimenta~ A01   STOMATOL~   18141
# i 67,586 more rows
# i 1 more variable: Cost <dbl>
```

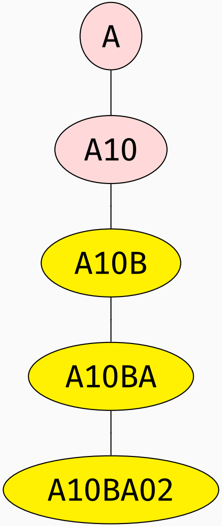
ATC drug classification

A	Alimentary tract and metabolism
B	Blood and blood forming organs
C	Cardiovascular system
D	Dermatologicals
G	Genito-urinary system and sex hormones
H	Systemic hormonal preparations, excluding sex hormones and insulins
J	Anti-infectives for systemic use
L	Antineoplastic and immunomodulating agents
M	Musculo-skeletal system
N	Nervous system
P	Antiparasitic products, insecticides and repellents
R	Respiratory system
S	Sensory organs
V	Various

ATC drug classification

ATC1: 14 classes

ATC2: 84 classes



Alimentary tract and metabolism

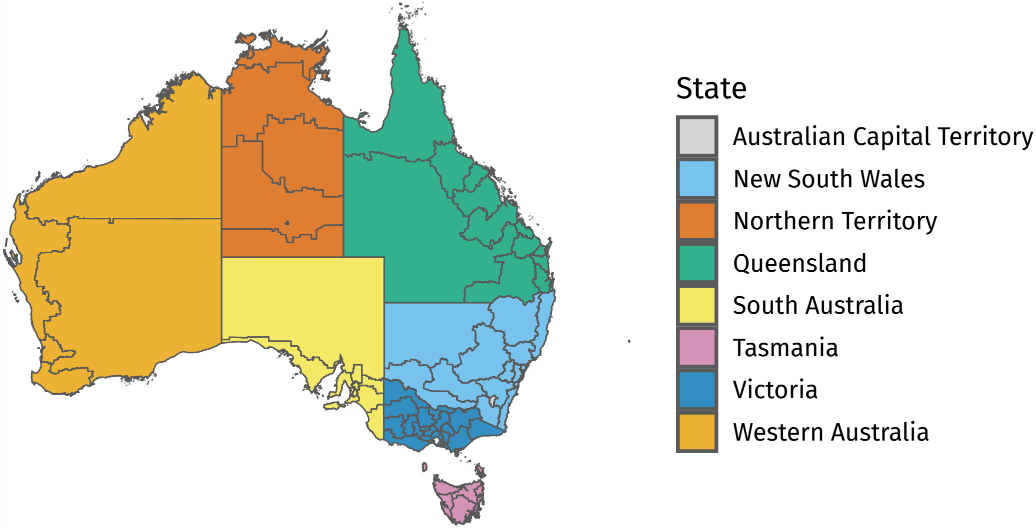
Drugs used in diabetes

Blood glucose lowering drugs

Biguanides

Metformin

Australian tourism



Australian tourism

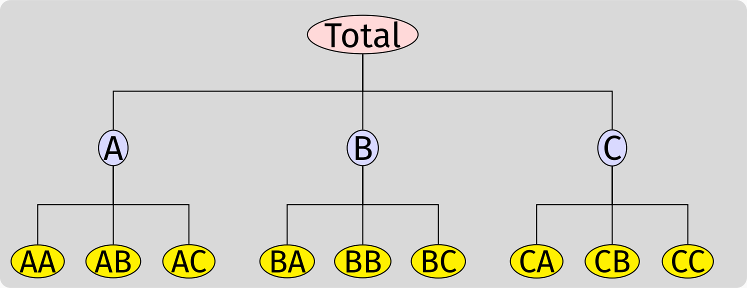
```
tourism
```

```
# A tsibble: 24,320 x 5 [1Q]
# Key:       Region, State, Purpose [304]
   Quarter Region   State Purpose  Trips
     <qtr> <chr>    <chr> <chr>    <dbl>
 1 1998 Q1 Adelaide SA    Business  135.
 2 1998 Q2 Adelaide SA    Business  110.
 3 1998 Q3 Adelaide SA    Business  166.
 4 1998 Q4 Adelaide SA    Business  127.
 5 1999 Q1 Adelaide SA    Business  137.
 6 1999 Q2 Adelaide SA    Business  200.
 7 1999 Q3 Adelaide SA    Business  169.
 8 1999 Q4 Adelaide SA    Business  134.
 9 2000 Q1 Adelaide SA    Business  154.
10 2000 Q2 Adelaide SA    Business  169.
# i 24,310 more rows
```

- Quarterly data on visitor nights, 1998:Q1 – 2017:Q4
- From: *National Visitor Survey*, based on annual interviews of 120,000 Australians aged 15+, collected by Tourism Research Australia.
- Split by 8 states and 76 regions
- Split by purpose of travel
  - Holiday
  - Visiting friends and relatives (VFR)
  - Business
  - Other
- 304 bottom-level series

Hierarchical time series

A **hierarchical time series** is a collection of several time series that are linked together in a hierarchical structure.



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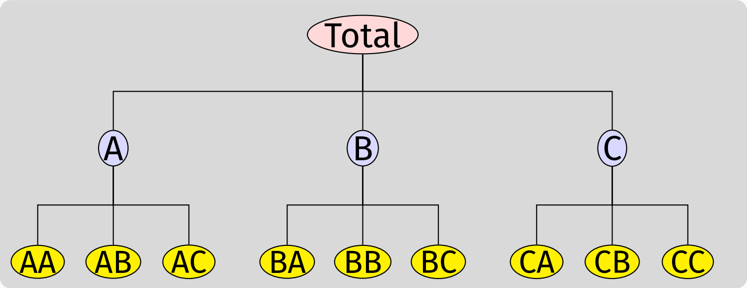
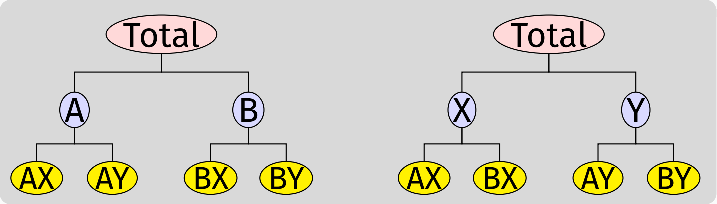


Examples

- PBS sales by ATC groups
- Tourism demand by states, regions

Grouped time series

A **grouped time series** is a collection of time series that can be grouped together in a number of non-hierarchical ways.



Grouped time series

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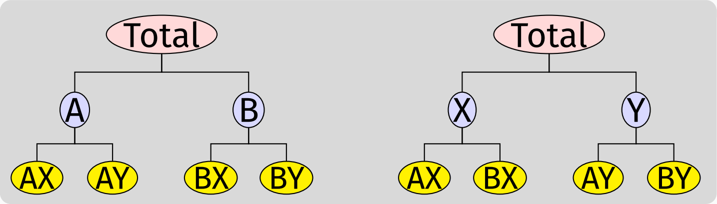

Examples

- Tourism by state and purpose of travel
- Retail sales by product groups/sub groups, and by countries/regions

Creating aggregates

```r
PBS |>
  aggregate_key(ATC1 / ATC2, Scripts = sum(Scripts)) |>
  filter(Month == yearmonth("1991 Jul")) |>
  print(n = 18)
```

```
# A tsibble: 98 x 4 [1M]
# Key:       ATC1, ATC2 [98]
      Month ATC1         ATC2         Scripts
      <mth> <chr*>       <chr*>         <dbl>
 1 1991 Jul <aggregated> <aggregated> 8090395
 2 1991 Jul A            <aggregated>  799025
 3 1991 Jul B            <aggregated>  109227
 4 1991 Jul C            <aggregated> 1794995
 5 1991 Jul D            <aggregated>  299779
 6 1991 Jul G            <aggregated>  300931
 7 1991 Jul H            <aggregated>  112114
 8 1991 Jul J            <aggregated> 1151681
 9 1991 Jul L            <aggregated>   24580
10 1991 Jul M            <aggregated>  562956
11 1991 Jul N            <aggregated> 1546023
12 1991 Jul P            <aggregated>   47661
13 1991 Jul R            <aggregated>  859273
14 1991 Jul S            <aggregated>  391639
15 1991 Jul V            <aggregated>   38705
16 1991 Jul Z            <aggregated>   51806
```

Creating aggregates

```r
tourism |>
  aggregate_key(Purpose * (State / Region), Trips = sum(Trips)) |>
  filter(Quarter == yearquarter("1998 Q1")) |>
  print(n = 15)
```

```
# A tsibble: 425 x 5 [1Q]
# Key:       Purpose, State, Region [425]
   Quarter Purpose      State        Region        Trips
     <qtr> <chr*>       <chr*>       <chr*>        <dbl>
 1 1998 Q1 <aggregated> <aggregated> <aggregated> 23182.
 2 1998 Q1 Business     <aggregated> <aggregated>  3599.
 3 1998 Q1 Holiday      <aggregated> <aggregated> 11806.
 4 1998 Q1 Other        <aggregated> <aggregated>   680.
 5 1998 Q1 Visiting     <aggregated> <aggregated>  7098.
 6 1998 Q1 <aggregated> ACT          <aggregated>   551.
 7 1998 Q1 <aggregated> NSW          <aggregated>  8040.
 8 1998 Q1 <aggregated> NT           <aggregated>   181.
 9 1998 Q1 <aggregated> QLD          <aggregated>  4041.
10 1998 Q1 <aggregated> SA           <aggregated>  1735.
11 1998 Q1 <aggregated> TAS          <aggregated>   982.
12 1998 Q1 <aggregated> VIC          <aggregated>  6010.
13 1998 Q1 <aggregated> WA           <aggregated>  1641.
14 1998 Q1 <aggregated> ACT          Canberra       551.
```

Creating aggregates

- Similar to `summarise()` but using the key structure
- A grouped structure is specified using `grp1 * grp2`
- A nested structure is specified via `parent / child`.
- Groups and nesting can be mixed:

```
(country/region/city) * (brand/product)
```

- All possible aggregates are produced.
- These are useful when forecasting at different levels of aggregation.

Outline

The problem

1. How to forecast time series at all nodes such that the forecasts add up in the same way as the original data?
2. Can we exploit relationships between the series to improve the forecasts?

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2. Can we exploit relationships between the series to improve the forecasts?

The solution

1. Forecast all series at all levels of aggregation using an automatic forecasting algorithm.
   (e.g., `ETS`, `ARIMA`, ...)
2. Reconcile the resulting forecasts so they add up correctly using least squares optimization (i.e., find closest reconciled forecasts to the original forecasts).
3. This is available using `reconcile()`.

Forecast reconciliation

```
tourism |>
  aggregate_key(Purpose * (State / Region), Trips = sum(Trips)) |>
  model(ets = ETS(Trips)) |>
  reconcile(ets_adjusted = min_trace(ets)) |>
  forecast(h = 2)
```

```
# A fable: 1,700 x 7 [1Q]
# Key:     Purpose, State, Region, .model [850]
   Purpose  State  Region          .model       Quarter        Trips .mean
   <chr*>   <chr*> <chr*>          <chr>          <qtr>       <dist> <dbl>
 1 Business ACT    Canberra        ets          2018 Q1 N(144, 1119) 144.
 2 Business ACT    Canberra        ets          2018 Q2 N(203, 2260) 203.
 3 Business ACT    Canberra        ets_adjusted 2018 Q1  N(157, 539) 157.
 4 Business ACT    Canberra        ets_adjusted 2018 Q2  N(214, 951) 214.
 5 Business ACT    <aggregated>    ets          2018 Q1 N(144, 1119) 144.
 6 Business ACT    <aggregated>    ets          2018 Q2 N(203, 2260) 203.
 7 Business ACT    <aggregated>    ets_adjusted 2018 Q1  N(157, 539) 157.
 8 Business ACT    <aggregated>    ets_adjusted 2018 Q2  N(214, 951) 214.
 9 Business NSW    Blue Mountains  ets          2018 Q1   N(20, 140)  19.7
```

Hierarchical and grouped time series

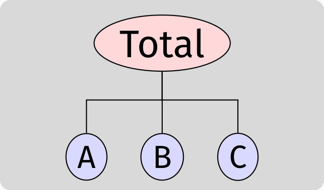
Every collection of time series with aggregation constraints can be written as

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

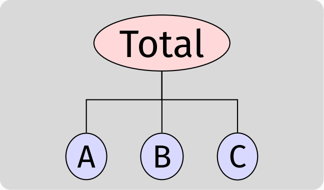
where

- $\boldsymbol{y}_t$ is a vector of all series at time t
- $\boldsymbol{b}_t$ is a vector of the most disaggregated series at time t
- $\boldsymbol{S}$ is a "summing matrix'' containing the aggregation constraints.

Hierarchical time series



Hierarchical time series

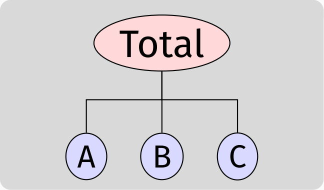


y_t : observed aggregate of all series at time t.

$y_{X,t}$: observation on series X at time t.

$\boldsymbol{b}_t$: vector of all series at bottom level in time t.

Hierarchical time series



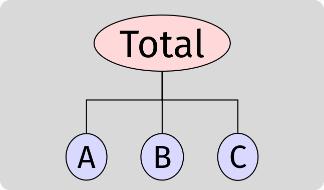
y_t : observed aggregate of all series at time t.

$y_{X,t}$: observation on series X at time t.

$\boldsymbol{b}_t$: vector of all series at bottom level in time t.

$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}$$

Hierarchical time series



y_t : observed aggregate of all series at time t.

$y_{X,t}$: observation on series X at time t.

$\boldsymbol{b}_t$: vector of all series at bottom level in time t.

$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

Forecasting notation

Let $\hat{\boldsymbol{y}}_n(h)$ be vector of initial h-step forecasts, made at time n, stacked in same order as $\boldsymbol{y}_t$.

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Reconciled forecasts must be of the form:
$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_n(h)$$
for some matrix $\boldsymbol{G}$.

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Reconciled forecasts must be of the form:

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_n(h)$$

for some matrix $\boldsymbol{G}$.

- $\boldsymbol{G}$ extracts and combines base forecasts $\hat{\boldsymbol{y}}_n(h)$ to get bottom-level forecasts.
- $\boldsymbol{S}$ adds them up

Optimal combination forecasts

Main result

The best (minimum sum of variances) unbiased forecasts are obtained when $\boldsymbol{G} = (\boldsymbol{S}'\Sigma_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\Sigma_h^{-1}$, where Σ_h is the h-step base forecast error covariance matrix.

Optimal combination forecasts

Main result

The best (minimum sum of variances) unbiased forecasts are obtained when $\boldsymbol{G} = (\boldsymbol{S}'\Sigma_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\Sigma_h^{-1}$, where Σ_h is the h-step base forecast error covariance matrix.

$$\tilde{\boldsymbol{y}}_n(h) = \boldsymbol{S}(\boldsymbol{S}'\Sigma_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\Sigma_h^{-1}\hat{\boldsymbol{y}}_n(h)$$

Problem: Σ_h hard to estimate, especially for $h > 1$.

Solutions:

- Ignore Σ_h (OLS) [`min_trace(method='ols')`]
- Assume $\Sigma_h = k_h\Sigma_1$ is diagonal (WLS) [`min_trace(method='wls')`]
- Assume $\Sigma_h = k_h\Sigma_1$ and estimate it (GLS) [`min_trace(method='shrink')` (the default)]

Features

- Covariates can be included in initial forecasts.
- Adjustments can be made to initial forecasts at any level.
- Very simple and flexible method. Can work with *any* hierarchical or grouped time series.
- Conceptually easy to implement: regression of base forecasts on structure matrix.

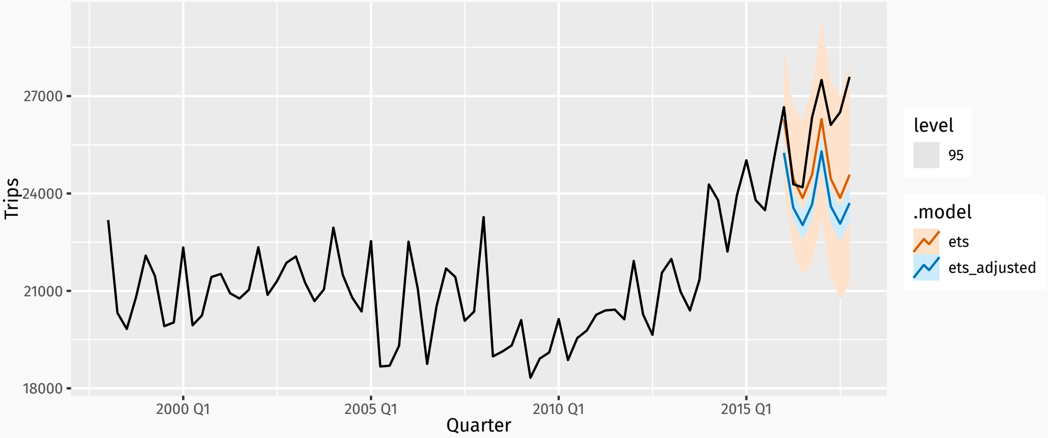
Outline

Example: Australian tourism

```r
tourism_agg <- tourism |>
  aggregate_key(Purpose * (State / Region),
    Trips = sum(Trips)
  )
fc <- tourism_agg |>
  filter_index(. ~ "2015 Q4") |>
  model(ets = ETS(Trips)) |>
  reconcile(ets_adjusted = min_trace(ets)) |>
  forecast(h = "2 years")
```

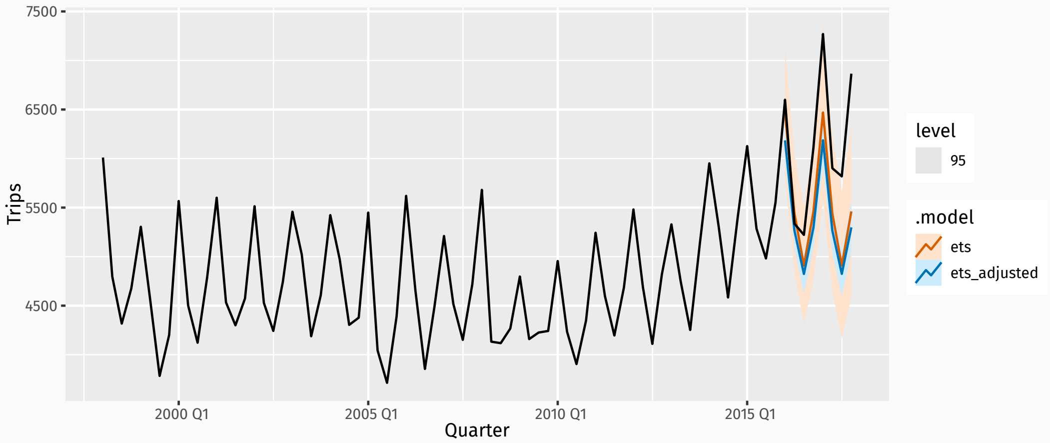
Example: Australian tourism

```
fc |>
  filter(is_aggregated(Purpose) & is_aggregated(State)) |>
  autoplot(tourism_agg, level = 95)
```



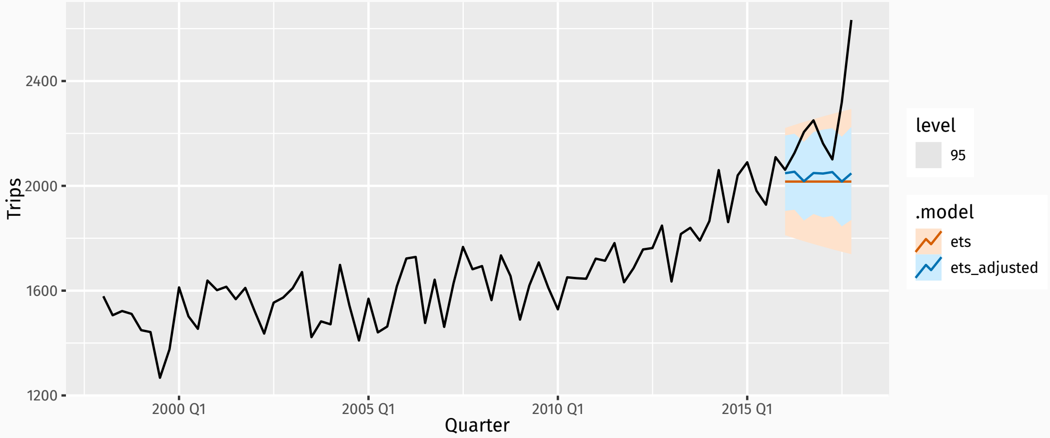
Example: Australian tourism

```
fc |>
  filter(is_aggregated(Purpose) & State == "VIC" & is_aggregated(Region)) |>
  autoplot(tourism_agg, level = 95)
```



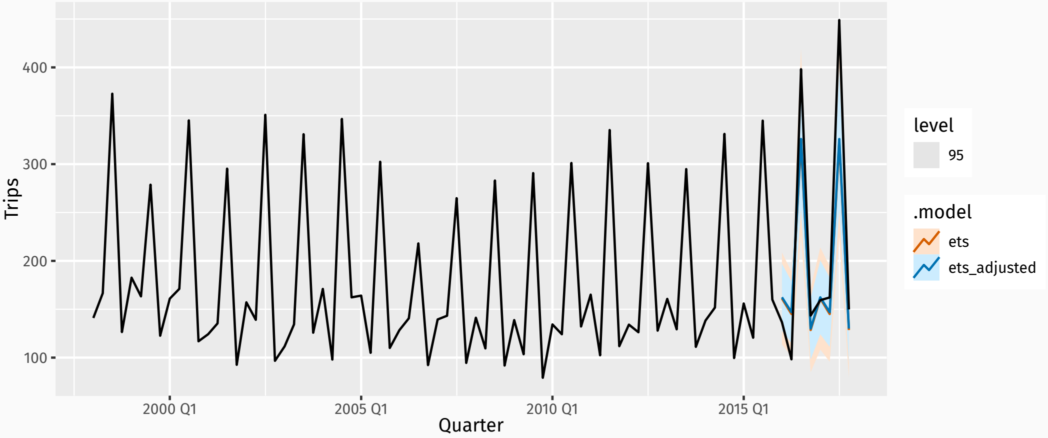
Example: Australian tourism

```
fc |>
  filter(is_aggregated(Purpose) & Region == "Melbourne") |>
  autoplot(tourism_agg, level = 95)
```



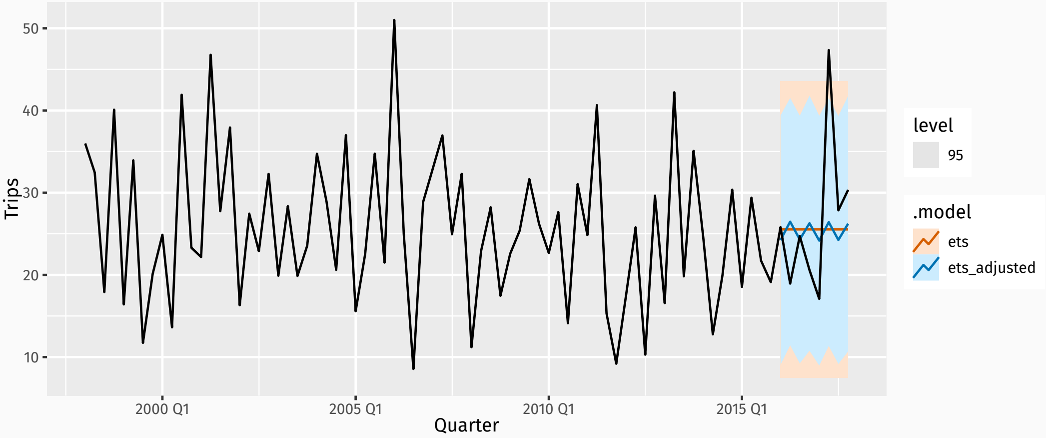
Example: Australian tourism

```
fc |>
  filter(is_aggregated(Purpose) & Region == "Snowy Mountains") |>
  autoplot(tourism_agg, level = 95)
```



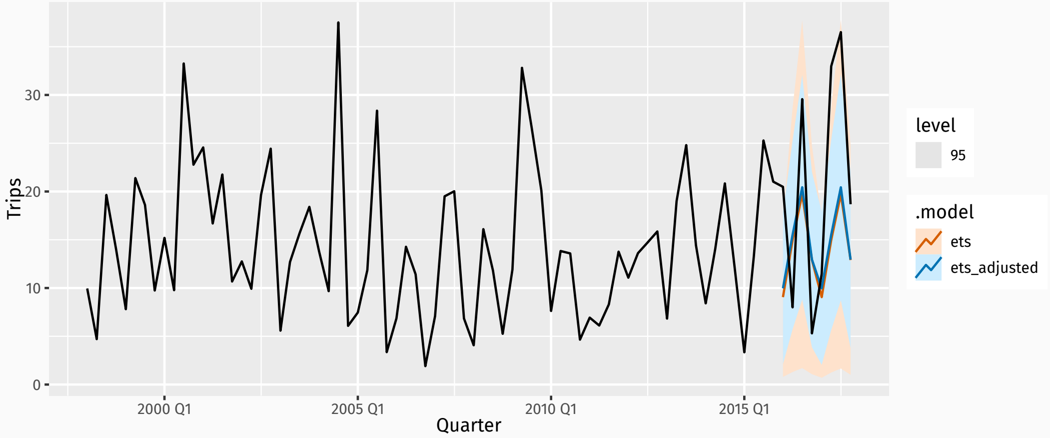
Example: Australian tourism

```r
fc |>
  filter(Purpose == "Holiday" & Region == "Barossa") |>
  autoplot(tourism_agg, level = 95)
```



Example: Australian tourism

```r
fc |>
  filter(is_aggregated(Purpose) & Region == "MacDonnell") |>
  autoplot(tourism_agg, level = 95)
```



Example: Australian tourism

```r
fc <- tourism_agg |>
  filter_index(. ~ "2015 Q4") |>
  model(
    ets = ETS(Trips),
    arima = ARIMA(Trips)
  ) |>
  mutate(
    comb = (ets + arima) / 2
  ) |>
  reconcile(
    ets_adj = min_trace(ets),
    arima_adj = min_trace(arima),
    comb_adj = min_trace(comb)
  ) |>
  forecast(h = "2 years")
```

Forecast evaluation

```r
fc |> accuracy(tourism_agg)
```

```
# A tibble: 2,550 x 13
   .model Purpose  State  Region                .type    ME  RMSE   MAE   MPE  MAPE  MASE
   <chr>  <chr*>   <chr*> <chr*>                <chr> <dbl> <dbl> <dbl> <dbl> <dbl> <dbl>
 1 arima  Business ACT    Canberra            ~ Test  35.9   45.7 35.9   16.9  16.9 0.938
 2 arima  Business ACT    <aggregated>          Test  35.9   45.7 35.9   16.9  16.9 0.938
 3 arima  Business NSW    Blue Mountains      ~ Test   1.93  10.6  8.52 -18.0  48.6 0.644
 4 arima  Business NSW    Capital Country     ~ Test   8.08  15.6 10.4   11.8  19.0 0.744
 5 arima  Business NSW    Central Coast       ~ Test  10.0   14.5 10.8   26.9  32.2 0.982
 6 arima  Business NSW    Central NSW         ~ Test  17.7   31.9 28.2   12.0  24.1 1.08
 7 arima  Business NSW    Hunter              ~ Test  35.3   43.9 35.3   24.2  24.2 1.30
 8 arima  Business NSW    New England North W~ Test  23.1   31.8 26.8   19.5  28.0 1.76
 9 arima  Business NSW    North Coast NSW     ~ Test  24.8   40.1 36.8   11.5  28.5 1.38
10 arima  Business NSW    Outback NSW         ~ Test   6.87  11.0  7.76  13.7  16.5 0.571
# i 2,540 more rows
# i 2 more variables: RMSSE <dbl>, ACF1 <dbl>
```

Forecast evaluation

```
fc |>
  accuracy(tourism_agg) |>
  group_by(.model) |>
  summarise(MASE = mean(MASE)) |>
  arrange(MASE)
```

```
# A tibble: 6 x 2
  .model     MASE
  <chr>     <dbl>
1 ets_adj    1.02
2 comb_adj   1.02
3 ets        1.04
4 comb       1.04
5 arima_adj  1.07
6 arima      1.09
```

Outline

Lab Session 20

- Prepare aggregations of the PBS data by Concession, Type, and ATC1.
- Use forecast reconciliation with the PBS data, using ETS, ARIMA and SNAIVE models, applied to all but the last 3 years of data.
- Which type of model works best?
- Does the reconciliation improve the forecast accuracy?
- Why doesn’t the reconcililation make any difference to the SNAIVE forecasts?